Functional Analysis

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Lecture 4

L^∞-spaces. Bounded operators.

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L^∞-space (essentially bounded functions)

Let (Ω, Σ, μ) be a fixed measure space.

Def. Measurable function $x : \Omega \to \mathbb{F}$ **essentially bounded**, if it is bounded outside a μ-null set. They form

$$L^\infty(\mu) := \{x : \Omega \to \mathbb{F}\,\text{measurable} : \exists_{A\in\Sigma, \mu(A)=0} \sup_{t\in\Omega\setminus A} |x(t)| < \infty\}$$

a linear space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, where

$$(x+y)(t) := x(t)+y(t), \quad (\lambda x)(t) := \lambda x(t) \qquad \left(\begin{array}{c}\text{pointwise}\\ \text{operations!}\end{array}\right)$$

$L^\infty(\mu)$ consist of functions with finite **essential supremum**

$$\|x\|_\infty := \inf_{\mu(A)=0} \sup_{t\in\Omega\setminus A} |x(t)| = \min_{\mu(A)=0} \sup_{t\in\Omega\setminus A} |x(t)|$$

Rem. $\|x\|_\infty = 0 \iff \mu(\{t \in \Omega : x(t) \neq 0\}) = 0 \overset{def}{\iff} x \overset{\mu\text{-a.e}}{=} 0$

Convention:

We identify functions in $L^\infty(\mu)$ which are equal μ-a.e.
(formally elements of $L^\infty(\mu)$ are equivalence classes for $y \overset{\mu\text{-a.e.}}{=} x$).
Then $(L^\infty(\mu), \|\cdot\|_\infty)$ is a normed space!

Thm. $L^\infty(\mu)$ with $\|\cdot\|_\infty$ is a Banach space.

Proof:

Rem. $x_n \xrightarrow{L^\infty} x \iff x_n|_{\Omega\setminus A} \rightrightarrows x|_{\Omega\setminus A}$ where $\mu(A) = 0,\ A \in \Sigma$

Ex. If μ is the counting measure then $L^\infty(\mu)$ becomes

- $B(\Omega) := \{x : \Omega \to \mathbb{F} : \sup_{t\in\Omega} |x(t)| < \infty\}$ with the norm $\|x\|_\infty := \sup_{t\in\Omega} |x(t)|$ – **space of bounded functions**
- ℓ^∞ with the norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x(k)| \qquad (\Omega = \mathbb{N})$
- $\mathbb{F}^n$ with the norm $\|x\|_\infty = \max_{k=1,\ldots,n} |x(k)| \qquad (\Omega = \{1, \ldots, n\})$

Classical Banach spaces (cheat sheet)

In the table below $p \in [1, \infty)$.

Symb.	**Banach space**	**norm**
$B(\Omega)$	bounded functions	$\|x\|_\infty = \sup_{t\in\Omega} \|x(t)\|$
$C_b(\Omega)$	continuous and bounded functions	$\|x\|_\infty = \sup_{t\in\Omega} \|x(t)\|$
$C_0(\Omega)$	continuous functions, vanishing at ∞	$\|x\|_\infty = \max_{t\in\Omega} \|x(t)\|$
$L^p(\mu)$	"functions"integrable in the p-th power	$\|x\|_p = \left(\int_\Omega \|x(t)\|^p \, d\mu\right)^{\frac{1}{p}}$
$L^\infty(\mu)$	essentially bounded "functions"	$\|x\|_\infty = \inf_{\mu(A)=0} \sup_{t\in\Omega\setminus A} \|x(t)\|$
ℓ^∞	bounded sequences	$\|x\|_\infty = \sup_{k\in\mathbb{N}} \|x(k)\|$
ℓ^p	sequences summable in the p-th power	$\|x\|_p = \left(\sum_{k=1}^{\infty} \|x(t)\|^p\right)^{\frac{1}{p}}$
c	convergent sequences	$\|x\|_\infty = \max_{k\in\mathbb{N}} \|x(k)\|$
c_0	sequences convergent to zero	$\|x\|_\infty = \max_{k\in\mathbb{N}} \|x(k)\|$

Bounded operators

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be fixed normed spaces.

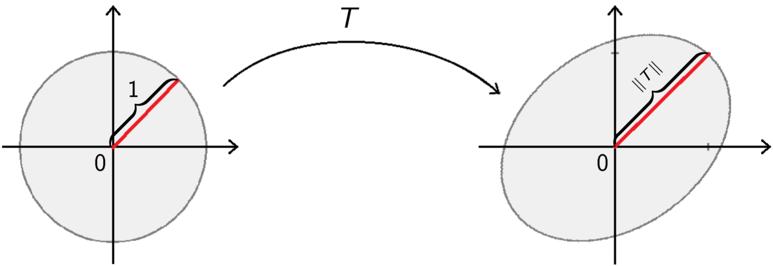
Def. Operators are linear maps $T : X \to Y$. We say that an operator T is **bounded** if

$$\exists_{C \geqslant 0}\ \forall_{x \in X} \quad \|Tx\| \leqslant C\|x\| \qquad \left(\begin{array}{c}\text{boundeness}\\ \text{inequality}\end{array}\right)$$

The **norm of the operator** T is the smallest C in the above inequality

$$\|T\| := \inf\{C : \|Tx\| \leqslant C\|x\| \text{ for every } x \in X\}$$

Prop. $\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$



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$$\|T\| := \inf\{C : \|Tx\| \leqslant C\|x\| \text{ for every } x \in X\}$$

Prop. $\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$

T is bounded $\iff$ $\|T\| < \infty$

Proof: Note that $\|Tx\| \leqslant \|T\| \cdot \|x\|$, for $x \in X$. Hence

$$C := \sup_{\|x\|=1} \|Tx\| \leqslant \sup_{\|x\| \leqslant 1} \|Tx\| \leqslant \|T\|. \qquad (\dagger)$$

Thus $\|T\| < \infty \implies C < \infty$. Converesly, if $C < \infty$ for $x \in X \setminus \{0\}$ we have $\|\frac{x}{\|x\|}\| = 1$ and $\|T\frac{x}{\|x\|}\| \leqslant C \iff \frac{\|T(x)\|}{\|x\|} \leqslant C \iff \|Tx\| \leqslant C\|x\|$.

Thus $\|T\| \leqslant C$. In particular, $(\dagger)$ implies the assertion. ■

Thm. Let $T : X \to Y$ be a linear operator. TFAE:

1. T is bounded
2. T is continuous
3. T is continuous at some point $x_0 \in X$
4. T is continuous at zero

Proof: (1)⇒(2). Bounded operator T is a Lipschitz map:

$$\|Tx - Ty\| = \|T(x-y)\| \leqslant \|T\| \|x-y\|.$$

Hence $x \to y \implies Tx \to Ty$. Accordingly, T is continuous.

(2)⇒(3). Clear (a map is continuous $\overset{def}{\iff}$ it is continuous at every point)

(3)⇒(4). If $x_n \to 0$, then $x_n + x_0 \to x_0$. Hence by continuity of T at x_0

$$Tx_n = T(x_n + x_0) - Tx_0 \longrightarrow Tx_0 - Tx_0 = 0 = T(0).$$

That means that T is continuous at zero.

(4)⇒(1). Assume T is unbounded. Then there is $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ and $\|Tx_n\| \geqslant n$, for $n \in \mathbb{N}$. Thereforfe $\|\frac{x_n}{\sqrt{n}}\| = \frac{1}{\sqrt{n}} \to 0$, so $\frac{x_n}{\sqrt{n}} \to 0$, and $\|T\frac{x_n}{\sqrt{n}}\| = \frac{1}{\sqrt{n}}\|Tx_n\| \geqslant \frac{1}{\sqrt{n}} \cdot n = \sqrt{n} \to \infty$. Hence T is discontinuous at zero. ■

Thm. (Continuous extension of bounded operators)

A bounded operator $T : X_0 \to Y$ defined on a dense subspace $X_0 \subseteq X$ with values in a Banach space Y **extends uniquely** to a bounded operator $\overline{T} : X \to Y$. Then $\|T\| = \|\overline{T}\|$.

Proof: Let $x_0 \in X = \overline{X_0}$. Take $\{x_n\}_{n=1}^{\infty} \subseteq X_0$ convergent to x_0. Then $\{Tx_n\}_{n=1}^{\infty} \subseteq Y$ is Cauchy in Y, as

$$\|Tx_n - Tx_m\| \leqslant \|T\| \cdot \|x_n - x_m\| \longrightarrow 0, \qquad \text{when } n, m \to \infty.$$

Since Y is complete, $\{Tx_n\}_{n=1}^{\infty}$ converges to some $y_0 \in Y$.
The limit y_0 does not depend on the choice of $\{x_n\}_{n=1}^{\infty}$.

Putting $\overline{T}x_0 := y_0$ we get a well-defined operator $\overline{T} : X \to Y$ that extends T. Moreover,

$$\|\overline{T}x_0\| = \lim_{n\to\infty} \|Tx_n\| \leqslant \lim_{n\to\infty} \|T\| \cdot \|x_n\| = \|T\| \cdot \|x_0\|.$$

Hence $\overline{T}$ is bounded and $\|\overline{T}\| \leqslant \|T\|$.
Inequality $\|T\| \leqslant \|\overline{T}\|$ is clear, because $\overline{T}$ extends T. ∎

Ex. 1 (Integration)

Let $X := L^1(\mu)$ and $Y := \mathbb{F}$. Then the **integral**

$$Tx := \int_\Omega x(t)\,d\mu(t), \qquad x \in L^1(\mu),$$

is a bounded operator $T : L^1(\mu) \to \mathbb{F}$ and $\|T\| = 1$:

$$|Tx| = \left|\int_\Omega x(t)\,d\mu(t)\right| \leqslant \int_\Omega |x(t)|\,d\mu(t) = 1 \cdot \|x\|_1,$$

whence $\|T\| \leqslant 1$. On the other hand, for any measurable $A \subseteq \Omega$ such that $0 < \mu(A) < \infty$ putting $x := \frac{1}{\mu(A)}\mathbb{1}_A$ we get

$$\int_\Omega x\,d\mu = \int_\Omega \frac{1}{\mu(A)}\mathbb{1}_A\,d\mu = 1,$$

that is $\|x\|_1 = 1$ and $|Tx| = 1$, so $1 \leqslant \|T\|$. Hence $\|T\| = 1$.

Prz. 2 (Differentiation)

Let $X := C^{(1)}([0,1])$ be **the space of continuously differentiable functions** and let $Y = C([0,1])$. Consider these spaces with the supremum norm. **Differentiation**

$$(Tx)(t) := x'(t) \qquad x \in C^{(1)}([0,1]), t \in [0,1],$$

yieds a well-defined operator $T : X \to Y$ which is **unbounded**! Indeed, for $x_n(t) = t^n$ we have

$$\|x_n\|_\infty = 1 \quad \text{oraz} \quad \|Tx_n\| = \|x_n'\|_\infty = \sup_{t\in[0,1]} |nt^{n-1}| = n \to \infty.$$

The standard norm on $C^{(1)}([0,1])$ is given by the formula

$$\|x\|_1 = \|x\|_\infty + \|x'\|_\infty = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|.$$

With this norm on X the operator T is bounded and $\|T\| = 1$.

Ex. 3 (Multiplication operator)

Let $X = Y = L^p(\mu)$, $1 \leqslant p < \infty$. **Multiplication** by $a \in L^\infty(\mu)$

$$(Tx)(t) := a(t)x(t), \qquad x \in L^p(\mu), t \in \Omega,$$

defines a bounded operator $T : L^p(\mu) \to L^p(\mu)$ and $\|T\| = \|a\|_\infty$.

Indeed, the inequality $\|T\| \leqslant \|a\|_\infty$ follows from

$$\begin{aligned} \|Tx\|_p &= \left(\int_\Omega |a(t)x(t)|^p \, d\mu\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_\Omega \|a\|_\infty^p \cdot |x(t)|^p \; d\mu\right)^{\frac{1}{p}} = \|a\|_\infty \cdot \|x\|_p. \end{aligned}$$

Put $A_n := \{t : |a(t)|^p \geqslant \|a\|_\infty^p - 1/n\}$ and $x_n := \frac{1}{\mu(A_n)^{1/p}} \mathbb{1}_{A_n}$.

Then $\|x_n\|_p = 1$ and

$$\begin{aligned} \|Tx_n\|_p^p &= \frac{1}{\mu(A_n)} \cdot \int_{A_n} |a(t)|^p \, d\mu \\ &\geqslant \frac{1}{\mu(A_n)} \cdot \int_{A_n} \|a\|_\infty^p - \frac{1}{n} \; d\mu = \|a\|_\infty^p - \frac{1}{n}. \end{aligned}$$

Hence $\|Tx_n\|_p \to \|a\|_\infty$ and so $\|T\| \geqslant \|a\|_\infty$.

Ex. 4 (Composition operator)

Let $X = Y = C(\Omega)$ for a comapct space Ω and let $\varphi : \Omega \to \Omega$ be a continous map. The **composition**

$$(Tx)(t) := x(\varphi(t)), \qquad x \in C(\Omega), t \in \Omega,$$

defines a bounded operator $T : C(\Omega) \to C(\Omega)$ with $\|T\| = 1$.

Indeed, the inequality $\|T\| \leqslant 1$ follows from

$$\begin{aligned} \|Tx\|_\infty = \ & \sup_{t\in\Omega} |x(\varphi(t))| = \sup_{s\in\varphi(\Omega)\subseteq\Omega} |x(s)| \\ & \leqslant \sup_{s\in\Omega} |x(s)| = \|x\|_\infty. \end{aligned}$$

Taking $x \equiv 1$ (function constantly equal to 1) we get $\|x\|_\infty = 1$ and $\|Tx\|_\infty = \|x\|_\infty = 1$. Hence $1 \leqslant \|T\|$. Thus $\|T\| = 1$.

Ex. 5 (Composition operator on L^p)

Let $X = Y = L^p[0,1]$, for $p \geqslant 1$ and $\varphi(t) = t^2$. Show that the **composition** $(Tx)(t) := x(\varphi(t))$ defines a bounded operator $T : L^p[0,1] \to L^p[0,1]$ with $\|T\| = 2^{\frac{1}{p}}$.